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ON 4-MANIFOLDS HOMOTOPY EQUIVALENT TO CIRCLE BUNDLES OVER 3-MANIFOLDS

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ABSTRACT

We give criteria for a closed 4-manifold to be homotopy equivalent to the total space of an S^1-bundle over a closed 3-manifold. In the aspherical case the conditions are that the Euler characteristic be 0 and that the fundamental group have an infinite cyclic normal subgroup such that the quotient group has one end and finite cohomological dimension. Under further assumptions on this quotient group we characterize the total spaces of such bundles over $\widetilde{SL}$- or $H^2 \times E^1$-manifolds and over E^3-, Nil^3- or Sol^3-manifolds up to s-cobordism and homeomorphism respectively.

Introduction

In this paper we shall give criteria for a closed 4-manifold M to be homotopy equivalent to the total space of an S^1-bundle with base a closed 3-manifold N with infinite fundamental group. If each irreducible factor of N is either Haken, homotopy equivalent to a hyperbolic 3-manifold or Seifert fibred and $\pi_1(M)$ is torsion free then any such homotopy equivalence is simple. We shall show that M is homotopy equivalent to the total space of an S^1-bundle over an aspherical PD_3-complex if and only if $\chi(M) = 0$ and $\mu = \pi_1(M)$ has an infinite cyclic normal subgroup A such that μ/A has one end and finite cohomological dimension. If moreover μ/A has a subgroup of finite index with infinite abelianization and with a nontrivial abelian normal subgroup B/A then M is s-cobordant to the total space E of an S^1-bundle over a 3-manifold which is Seifert fibred or is a Sol^3-manifold; if B/A has rank at least 2 then M is in fact homeomorphic to E.

This note is loosely related to two other papers on PD-fibrations of 4-manifolds over S^1 and over surfaces ([12] and [13], respectively).

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1. The Homotopy Type of Certain 4-Manifolds

If M is a manifold or cell complex $P_2(M)$ shall denote the second stage of the Postnikov tower for M, and $c_M = g_M f_M$ the factorization of the classifying map $c_M : M \to K(\pi_1(M),1)$ through $f_M : M \to P_2(M)$ and $g_M : P_2(M) \to K(\pi_1(M),1)$. A map $f : X \to K(\pi_1(M),1)$ lifts to a map from X to $P_2(M)$ if and only if $f^* k_1 = 0$, where k_1 is the first Postnikov invariant of M in $H^3(\pi_1(M);\pi_2(M))$. The set of self maps k of $P_2(M)$ such that $g_M k = g_M$ acts transitively on the set of all such lifts. Note that $P_2(M) = L(\tilde{\pi}_2(M),2)$, the analogue of a $K(\pi_2(M),2)$-space in the category of spaces over $K(\pi_1(M),1)$, in the notation of page 300 of [1]. (This reference gives a detailed treatment of Postnikov factorizations of nonsimple maps and spaces.)

If w is a homomorphism from a group G to $Z/2Z$ we shall let Z^w denote the G-module with underlying abelian group Z and G-action given by $g.n = (-1)^{w(g)}n$ for all g in G and n in Z. If M is a closed m-dimensional manifold or Poincaré duality complex $[M]$ shall denote a generator of $H_m(M; Z^{w_1(M)})$, where the first Stiefel–Whitney class $w_1(M)$ is considered as a homomorphism from $\pi_1(M)$ to $Z/2Z$.

THEOREM 1: *Let E be a closed 4-manifold with fundamental group μ and suppose that f_E induces a monomorphism from $H_4(E; Z^{w_1(E)})$ to $H_4(P_2(E); Z^{w_1(E)})$ A closed 4-manifold M is homotopy equivalent to E if and only if there is an isomorphism θ from $\pi_1(M)$ to μ such that $w_1(M) = w_1(E)\theta$, there is a lift $\hat{c} : M \to P_2(E)$ of θc_M such that $\hat{c}_*[M] = \pm f_{E*}[E]$ and $\chi(M) = \chi(E)$.*

Proof: The conditions are clearly necessary. Conversely, suppose that they hold. We shall adapt to our situation the arguments of Hendriks [8] in analyzing the obstructions to the existence of a degree 1 map between PD_3-complexes realizing a given homomorphism of fundamental groups. For simplicity of notation we shall write $\tilde{Z}$ for $Z^{w_1(E)}$ and also for $Z^{w_1(M)}(= \theta^*\tilde{Z})$, and use θ to identify $\pi_1(M)$ with μ and $K(\pi_1(M),1)$ with $K(\mu,1)$. We may suppose the fundamental class $[M]$ is so chosen that $\theta_* c_{M*}[M] = c_{E*}[E]$. Then $\hat{c}_*[M] = f_{E*}[E]$.

Let $E_o = E \backslash D^4$. Then $P_2(E_o) = P_2(E)$ and may be constructed as the union of E_o with cells of dimension ≥ 4. Let $h : \tilde{Z} \otimes_\mu \pi_4(P_2(E_o), E_o) \to H_4(P_2(E_o), E_o; \tilde{Z})$ be the $w_1(E)$-twisted relative Hurewicz homomorphism, and let ∂ be the connecting homomorphism from $\pi_4(P_2(E_o), E_o)$ to $\pi_3(E_o)$ in the exact sequence of homotopy for the pair $(P_2(E_o), E_o)$. Then h and ∂ are isomorphisms since f_{E_o} is 3-connected. The composite of the inclusion $H_4(P_2(E); \tilde{Z}) = H_4(P_2(E_o); \tilde{Z}) \to H_4(P_2(E_o), E_o; \tilde{Z})$ with h^{-1} and $1 \otimes_\mu \partial$ gives a monomorphism τ_E from $H_4(P_2(E); \tilde{Z})$ to $\tilde{Z} \otimes_\mu \pi_3(E_o)$. Similarly $M_o = M \backslash D^4$ may be viewed as a subspace of $P_2(M_o)$ and there is a monomorphism τ_M from $H_4(P_2(M); \tilde{Z})$ to $\tilde{Z} \otimes_\mu \pi_3(M_o)$. These monomorphisms are natural with respect to maps defined on the 3-skeleta of the spaces (i.e., E_o and M_o).

The classes $\tau_E(f_{E*}[E])$ and $\tau_M(f_{M*}[M])$ are the images of the primary obstructions to retracting E onto E_o and M onto M_o, under the Poincaré duality isomorphisms from $H^4(E, E_o; \pi_3(E_o))$ to $H_0(E \backslash E_o; \tilde{Z} \otimes_\mu \pi_3(E_o)) = \tilde{Z} \otimes_\mu \pi_3(E_o)$ and from $H^4(M, M_o; \pi_3(M_o))$ to $\tilde{Z} \otimes_\mu \pi_3(M_o)$, respectively. Since M_o is homotopy equivalent to a cell complex of dimension ≤ 3 the restriction of $\hat{c}$ to M_o is homotopic to a map from M_o to E_o. In particular, $(1 \otimes_\mu \hat{c}_\sharp)\tau_M(f_{M*}[M]) = \tau_E(f_{E*}[E])$, where $\hat{c}_\sharp$ is the homomorphism from $\pi_3(M_o)$ to $\pi_3(E_o)$ induced by $\hat{c}|M_o$. It follows as in [8] that the obstruction to extending $\hat{c}|M_o : M_o \to E_o$ to a map d from M to E is trivial.

Since $f_{E*}d_*[M] = \hat{c}_*[M] = f_{E*}[E]$ and f_{E*} is a monomorphism in degree 4 the map d has degree 1. The only obstruction to d being a homotopy equivalence is the "surgery kernel" ker $\pi_2(d)$, which is a stably free $Z[\mu]$-module, by Lemma 2.3 of [20]. Up to homotopy type we may assume that E and M are finite cell complexes, and that M is a subcomplex of E. On counting bases for the equivariant cellular chain complex of the universal covering of the pair (E, M) we find that $(\ker \pi_2(d)) \oplus Z[\mu]^a = Z[\mu]^b$ for some nonnegative integers a and b such that $a + \chi(M) = b + \chi(E)$ (cf. the argument of Theorems 3 and 7 of Chapter 3 of [11]). Since $\chi(M) = \chi(E)$ we have $a = b$. It now follows from a lemma of Kaplansky that ker $\pi_2(d) = 0$. (Kaplansky did not publish his lemma in detail, but a complete proof of a generalization may be found in [15].) Thus d is a homotopy equivalence. □

If there is such a lift $\hat{c}$ then $c_M^* \theta^* k_1 = 0$ and $\theta_* c_{M*}[M] = c_{E*}[E]$. It can be shown that these conditions imply that there is a lift $\hat{c}$ such that $\hat{c}_*[M] - f_{E*}[E]$ lies in the image of $H_4(K(\pi_2(E), 2); Z)$ in $H_4(P_2(E); \tilde{Z})$.

2. Bundles over 3-Manifolds

If $p : E \to B$ is the projection of an S^1-bundle ξ over a connected base B then the natural map $p_* : \pi_1(E) \to \pi_1(B)$ is an epimorphism with cyclic kernel. The action of $\pi_1(B)$ on ker p_* determined by conjugation in $\pi_1(E)$ is given by

$w = w_1(\xi) : \pi_1(B) \to Z/2Z \cong \{\pm 1\} \leq \mathrm{Aut}(\ker\ p_*)$. For if α is any loop in B the total space of the induced bundle $\alpha^*\xi$ is the torus if $w(\alpha) = 0$ and the Klein bottle if $w(\alpha) = 1$ in $Z/2Z$; hence $gzg^{-1} = z^{\epsilon(g)}$ where $\epsilon(g) = (-1)^{w(p_*(g))}$ for g in $\pi_1(E)$ and z in $\ker\ p_*$. If the base B is a manifold we may use naturality and the Whitney sum formula (applied to the associated R^2-bundle $\hat{\xi}$) to show that $w_1(E) = p^*(w_1(B) + w_1(\xi))$. (As $p^* : H^1(B; Z/2Z) \to H^1(E; Z/2Z)$ is a monomorphism this equation determines $w_1(\xi)$.)

Bundles for which $\ker\ p_* \cong Z$ have the following equivalent characterizations.

Lemma 1: *Let ξ be an S^1-bundle with total space E, base B and projection $p : E \to B$, and suppose that B is connected. Then the following conditions are equivalent:*

(i) *ξ is induced from an S^1-bundle over $K(\pi_1(B), 1)$ via c_B;*

(ii) *for each map $\beta : S^2 \to B$ the induced bundle $\beta^*\xi$ is trivial; and*

(iii) *the kernel of the epimorphism $p_* : \pi_1(E) \to \pi_1(B)$ induced by p is infinite cyclic.*

If these conditions hold then $c(\xi) = c_B^*\Xi$, where $c(\xi)$ is the characteristic class of ξ in $H^2(B; Z^w)$ and Ξ is the class of the extension of fundamental groups in $H^2(\pi_1(B); Z^w) = H^2(K(\pi_1(B), 1); Z^w)$, where $w = w_1(\xi)$.

Proof: Condition (i) implies condition (ii) as for any such map β the composite $c_B\beta$ is nullhomotopic. Conversely, as we may construct $K(\pi_1(B), 1)$ by adjoining cells of dimension ≥ 3 to B condition (ii) implies that we may extend ξ over the 3-cells, and as S^1-bundles over S^n are trivial for all $n > 2$ we may then extend ξ over the whole of $K(\pi_1(B), 1)$, so that (ii) implies (i). The equivalence of (ii) and (iii) follows on observing that (iii) holds if and only if $\partial\beta = 0$ for all such β, where ∂ is the connecting map from $\pi_2(B)$ to $\pi_1(S^1)$ in the exact sequence of homotopy for ξ, and on comparing this with the corresponding sequence for $\beta^*\xi$.

As the natural map from the set of S^1-bundles over $K(\pi, 1)$ with $w_1 = w$ (which are classified by $H^2(K(\pi, 1); Z^w)$) to the set of extensions of π by Z with π acting via w (which are classified by $H^2(\pi; Z^w)$) which sends a bundle to the extension of fundamental groups is an isomorphism we have $c(\xi) = c_B^*(\Xi)$. □

If N is a closed 3-manifold which has no summands of type $S^1 \times S^2$ or $S^1 \tilde{\times} S^2$ (i.e., if $\pi_1(N)$ has no infinite cyclic free factor) then every S^1-bundle over N with $w = 0$ restricts to a trivial bundle over any map from S^2 to N. For if ξ is such a bundle, with characteristic class $c(\chi)$ in $H^2(N; Z)$, and $\beta : S^2 \to N$ is any map then $\beta_*(c(\beta^*\xi)\cap[S^2]) = \beta_*(\beta^*c(\xi)\cap[S^2]) = c(\xi)\cap\beta_*[S^2] = 0$, as the Hurewicz homomorphism is trivial for such N. Since β_* is an isomorphism in degree 0 it follows that $c(\beta^*\xi) = 0$ and so $\beta^*\xi$ is trivial. (A similar argument applies for bundles with $w \neq 0$, provided the induced 2-fold covering space N^w has no summands of type $S^1 \times S^2$ or $S^1 \tilde{\times} S^2$.)

On the other hand the bundle with total space $S^1 \times S^3$, base $S^1 \times S^2$ and projection $\mathrm{id}_{S^1} \times \eta$ (where η is the Hopf fibration) is clearly nontrivial when pulled back over any essential map from S^2 to $S^1 \times S^2$, and is not induced from any bundle over $K(Z,1)$. Moreover, a closed 3-manifold N with no summands of type $S^1 \times S^2$ or $S^1 \tilde{\times} S^2$ may have a 2-fold covering space (corresponding to a homomorphism w from $\pi_1(N)$ to $Z/2Z$) with such summands; for instance $S^1 \times S^2$ is a 2-fold covering space of $RP^3 \sharp RP^3$.

In general, any S^1-bundle over B is induced from some bundle over $P_2(B)$. Given an epimorphism $\gamma : \mu \to \nu$ with cyclic kernel and such that the action of ν on $\ker\gamma$ determined by conjugation in μ factors through multiplication by ± 1 there is an S^1-bundle $p(\gamma) : X(\gamma) \to Y(\gamma)$ whose fundamental group sequence realizes γ and which is universal for such bundles; the total space of this universal bundle is a $K(\mu, 1)$ space (cf. Proposition 11.4 of [20]). If $\ker\gamma$ is finite then the base space $Y(\gamma)$ is an $L(Z^w, 2)$ space in the notation of [1].

THEOREM 2: *Let M be a closed 4-manifold and N a closed 3-manifold with infinite fundamental group. Then M is homotopy equivalent to the total space of an S^1-bundle over N if and only if*
(i) *there is an epimorphism γ from $\mu = \pi_1(M)$ to $\nu = \pi_1(N)$ with cyclic kernel, and the action of ν on $\ker\gamma$ induced by conjugation in μ factors through a homomorphism $w : \nu \to Z/2Z \cong \{\pm 1\} \leq \mathrm{Aut}(\ker\gamma)$;*
(ii) $w_1(M) = (w_1(N) + w)\gamma$;
(iii) *there are maps $y : P_2(N) \to Y(\gamma)$ and $\hat{c} : M \to P_2(N)$ such that $g_N = c_{Y(\gamma)}y$ and $y\hat{c} = p(\gamma)c_M$ and $(\hat{c}, c_M)_*[M] = \pm\mathrm{tr}(f_{N*}[N])$ in $H_4(P_2(N) \times_{Y(\gamma)} K(\mu,1); Z^{w_1(M)})$, where tr is the transgression; and*
(iv) $\chi(M) = 0$.

Proof: Since these conditions are homotopy invariant and hold if M is the total space of such a bundle, they are clearly necessary. Suppose conversely that they hold. Let $E = N \times_{K(\nu,1)} K(\mu, 1)$ and $P = P_2(N) \times_{Y(\gamma)} K(\mu, 1)$ be the total spaces of the S^1-bundles over N and $P_2(N)$ obtained by pulling back the universal S^1-bundle over $Y(\gamma)$ with fundamental group sequence as in condition (i) via the maps yg_N and y, respectively. Then $P \simeq P_2(E)$ and the natural map from E to P covering g_N may be identified with g_E. Moreover $w_1(E) = (w_1(N) + w)\gamma = w_1(M)$, as maps from μ to $Z/2Z$, and $\chi(E) = 0 = \chi(M)$, by conditions (ii) and (iv). Since ν is infinite the universal covering space of N is an open 3-manifold, and so any map from S^3 to N is trivial on homology. As $P_2(N)$ may be constructed as the union of N with cells of dimension ≥ 4 it follows that $H_3(f_N; Z^{w_1(N)})$ is a monomorphism. As the maps f_N and f_E induce a homomorphism between the homology spectral sequences of these S^1-bundles it follows easily that $H_4(f_E; Z^{w_1(E)})$ is also a monomorphism. Since condition (iii) gives us a map $(\hat{c}, c_M)$ from M to $P_2(E)$ such that $(\hat{c}, c_M)_*[M] = f_{E*}[E]$ Theorem 1 now implies that M is homotopy equivalent to E. □

When $\ker\gamma \cong Z$ the base $Y(\gamma)$ of the universal bundle is a $K(\nu, 1)$ space and $y = g_N$. Condition (iii) then implies that

(iiia) $c_M^*\gamma^*k_1 = 0$, where k_1 is the first Postnikov invariant of N; and

(iiib) $c_{M*}[M] = \pm G(c_{N*}[N])$, where G is the ("Gysin") homomorphism from $H_3(\nu; Z^{w_N}) = H_3(\nu; H_1(ker\gamma; Z^{w_M}))$ to $H_4(\mu; Z^{w_M})$ determined by the LHS spectral sequence for μ as an extension of ν by Z.

(Note that $H_1(\ker\gamma; Z^{w_M})$ is isomorphic to Z^{w_N} by (ii)). If $\nu = \pi_1(N)$ is torsion free then G is in fact an isomorphism, for $c.d.\nu \leq 3$ and so the LHS spectral sequence is trivial outside the block $0 \leq p \leq 3$, $0 \leq q \leq 1$.) If moreover ν is not free (i.e., if N has at least one aspherical summand) then $H_3(c_N; Z^{w_1(N)})$ is a monomorphism. Do these conditions in turn imply that (iii) holds?

If ν is finite then $\pi_2(N) = 0$, so every S^1-bundle over N is induced from a bundle over $K(\nu, 1)$. Conditions (i)-(iv) remain necessary, but it is no longer clear that there should be a degree 1 map from M to the total space of such a bundle.

Although Theorem 2 may be applied to characterize products $N \times S^1$ up to homotopy type, a different argument shows that the conditions on the fundamental group and Euler characteristic which are obviously necessary are almost sufficient alone to characterize orientable products.

THEOREM 3: *Let M be a closed orientable 4-manifold with $\chi(M) = 0$ and such that $\pi_1(M)$ is a direct product $\nu \times Z$. Then the covering space M' with fundamental group ν is an orientable PD_3-complex. If ν is torsion free and is the fundamental group of a closed orientable 3-manifold N then M is homotopy equivalent to $N \times S^1$ if and only if $w_2(M) = 0$.*

Proof: The first assertion is essentially Theorem 3 of Chapter 7 of [11]. Let ϕ be a generator of the covering group $\mathrm{Aut}(M/M') \cong Z$. If $\nu \cong \pi_1(M')$ is torsion free and is the fundamental group of a closed orientable 3-manifold N then the indecomposable free factors of ν are either the groups of aspherical closed 3-manifolds or infinite cyclic, and so by [18] and the unique factorization theorem for orientable 3-manifolds there is a homotopy equivalence $h : M' \to N$. The manifold M may be recovered up to homotopy equivalence as the mapping torus $M(\psi)$, where ψ is the self homotopy equivalence of N defined by $\psi = h\phi h^{-1}$. We may assume that ψ fixes a basepoint. Since $\pi_1(M) \cong \nu \times Z$ the map ψ induces the identity on $\pi_1(N)$, and so is homotopic to a rotation about a 2-sphere [9]. If $w_2(M) = 0$ then the rotation is homotopic to the identity and so M is homotopy equivalent to $N \times S^1$; the converse is clear. □

Let τ be the twist map of $S^1 \times S^2$, given by $\tau(x, y) = (x, \rho(x)(y))$ for all (x, y) in $S^1 \times S^2$, where ρ is an essential map from S^1 to $SO(3)$. The mapping torus $M(\tau)$ has fundamental group $\mathbb{Z} \times \mathbb{Z}$ and Euler characteristic 0 but $w_2(M(\tau)) \neq 0$ and $M(\tau)$ is not homotopy equivalent to a product. (Clearly however $M(\tau^2) = S^1 \times S^2 \times S^1$.)

Every PD_3-complex with torsion free fundamental group is a connected sum of aspherical PD_3-complexes and copies of $S^1 \times S^2$ or $S^1 \tilde{\times} S^2$, by Theorem C of [18]. Thus every such complex is homotopy equivalent to a closed 3-manifold if every PD_3-group is the fundamental group of some aspherical closed 3-manifold. (Some restriction on torsion is necessary—cf. [17]).

It would be of interest to have a theorem that involved only assumptions on M, without specifying N in advance. For instance, the conditions "$\chi(M) = 0$" and "$\mu = \pi_1(M)$ has an infinite cyclic normal subgroup A such that μ/A is virtually of finite cohomological dimension" are certainly necessary for M to PD-fibre over some 3-manifold. Are they sufficient? In the aspherical case there is such a characterization, modulo the question of whether every PD_3-group is a 3-manifold group.

THEOREM 4: *A closed 4-manifold M is homotopy equivalent to the total space of an S^1-bundle over an aspherical PD_3-complex if and only if $\chi(M) = 0$ and $\mu = \pi_1(M)$ has an infinite cyclic normal subgroup A such that μ/A has one end and finite cohomological dimension.*

Proof: The conditions are clearly necessary. Conversely, suppose that they hold. Since μ/A has one end $H^s(\mu/A; Z[\mu/A]) = 0$ for $s \leq 1$ and so $H^t(\mu; Z[\mu]) = 0$ for $t \leq 2$, by an LHS spectral sequence calculation. Therefore M is aspherical, by Theorem 3 of Chapter 3 of [11], and so μ is a PD_4-group. Since A is FP_∞ and $c.d.\mu/A < \infty$ the quotient μ/A is a PD_3-group, by Theorem 9.11 of [2].

Therefore M is homotopy equivalent to the total space of an S^1-bundle over the PD_3-complex $K(\mu/A, 1)$. □

Note that a finitely generated torsion free group has one end if and only if it is indecomposable as a free product and is neither infinite cyclic nor trivial.

In particular, M is homotopy equivalent to a product of an aspherical PD_3-complex with S^1 if and only if $\chi(M) = 0$ and $\mu \cong \nu \times Z$ where ν has one end and finite cohomological dimension.

3. Simple Homotopy Type and s-Cobordism

By imposing further conditions on the fundamental group μ (or on the base N) we can obtain stronger results. We shall say that a 3-manifold is homotopy-hyperbolic if it is homotopy equivalent to a 3-manifold with a hyperbolic structure. Every virtually Haken 3-manifold is either Haken, homotopy-hyperbolic or Seifert-fibred [4]. (It is an open question whether every irreducible 3-manifold with infinite fundamental group is virtually Haken.)

LEMMA 2: *Let μ be a group with an infinite cyclic normal subgroup A such that $\nu = \mu/A$ is torsion free and is a free product $\nu = *_{1\leq i\leq n}\nu_i$ where each factor is the fundamental group of an irreducible 3-manifold which is Haken, homotopy-hyperbolic or Seifert fibred. Then $Wh(\mu) = Wh(\nu) = 0$.*

Proof: (Note that our hypotheses allow the possibility that some of the factors ν_i are infinite cyclic.) Let μ_i be the preimage of ν_i in μ, for $1 \leq i \leq n$. Then μ is the generalized free product of the μ_i's, amalgamated over infinite cyclic subgroups. For all $1 \leq i \leq n$ we have $Wh(\mu_i) = 0$, by Lemma 1.2 of [16] if $K(\nu_i, 1)$ is Haken, by the main result of [5] if it is homotopy-hyperbolic, by an easy extension of Proposition 2.5 of [14] if it is Seifert fibred and by Theorem 19.5 of [19] if ν_i is infinite cyclic. As the group rings $Z[\mu_i]$ have finite cohomological dimension and the amalgamations are over subgroups whose group rings are regular noetherian, the Nil groups of Waldhausen for these amalgamations vanish, and so his Mayer-Vietoris sequence for the K-theory of group rings [19] gives us $Wh(\mu) = Wh(\nu) = 0$ also. □

This lemma may be used to strengthen Theorem 2 to give criteria for a closed 4-manifold M to be **simple** homotopy equivalent to the total space of an S^1-bundle, if the irreducible summands of the base N are all virtually Haken and $\pi_1(M)$ is torsion free.

THEOREM 5: *A closed 4-manifold M is s-cobordant to the total space of an S^1-bundle over an aspherical closed 3-manifold which is Seifert fibred or admits a Sol^3-structure if and only if $\chi(M) = 0$ and $\mu = \pi_1(M)$ has normal subgroups $A < B < C$ such that A is infinite cyclic, μ/A has one end and finite cohomological dimension, B/A is nontrivial abelian, C has finite index in μ and C/A has infinite abelianization. If B/A has rank at least 2 then M is homeomorphic to such a bundle space.*

Proof: The conditions are clearly necessary. If they hold then μ/A is a PD_3-group by Theorem 4 and has a subgroup of finite index with a nontrivial abelian normal subgroup and infinite abelianization. Therefore μ/A is the group of a closed Seifert fibred 3-manifold (if there is such a subgroup B with B/A of rank 1 or 3) or of a closed 3-manifold with a Sol-structure (if the rank of B/A is 2, for every such subgroup B) [10]. Hence M is homotopy equivalent to the total space E of such a bundle. Any such homotopy equivalence must be simple, by Lemma 2. Since the surgery obstruction maps θ_4 from $[M; G/TOP]$ to $L_4^s(\mu; w_M)$ and θ_5 from $[SM; G/TOP]$ to $L_5^s(\mu; w_M)$ are isomorphisms [14], M is in fact s-cobordant to E. Finally if there is such a subgroup B such that B/A has rank at least 2 (corresponding to the base having an E^3-, Nil^3- or Sol^3-structure) then μ is virtually poly-Z (and is torsion free), so 5-dimensional s-cobordisms with group μ are products [7]. □

If a closed 4-manifold M with fundamental group μ is s-cobordant to the total space of an S^1-bundle over a homotopy-hyperbolic 3-manifold then $\chi(M) = 0$ and μ has an infinite cyclic normal subgroup A such that μ/A has one end and finite cohomological dimension and has no noncyclic abelian subgroup. We could use Theorem 10.7 of [6] instead of [14] to prove the converse if every PD_3-group is a 3-manifold group and if the geometrization conjecture for atoroidal 3-manifolds is true. Similarly we may show that simple homotopy equivalence implies s-cobordism when the base is Haken with square root closed accessible fundamental group, using [3] instead of [14]. However we do not yet have good intrinsic characterizations of the fundamental groups of such 3-manifolds comparable to the result of [10]. (The papers [3], [6] and [14] do not explicitly consider the 4-dimensional cases. However their results remain valid in dimension 4 up to s-cobordism.)

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